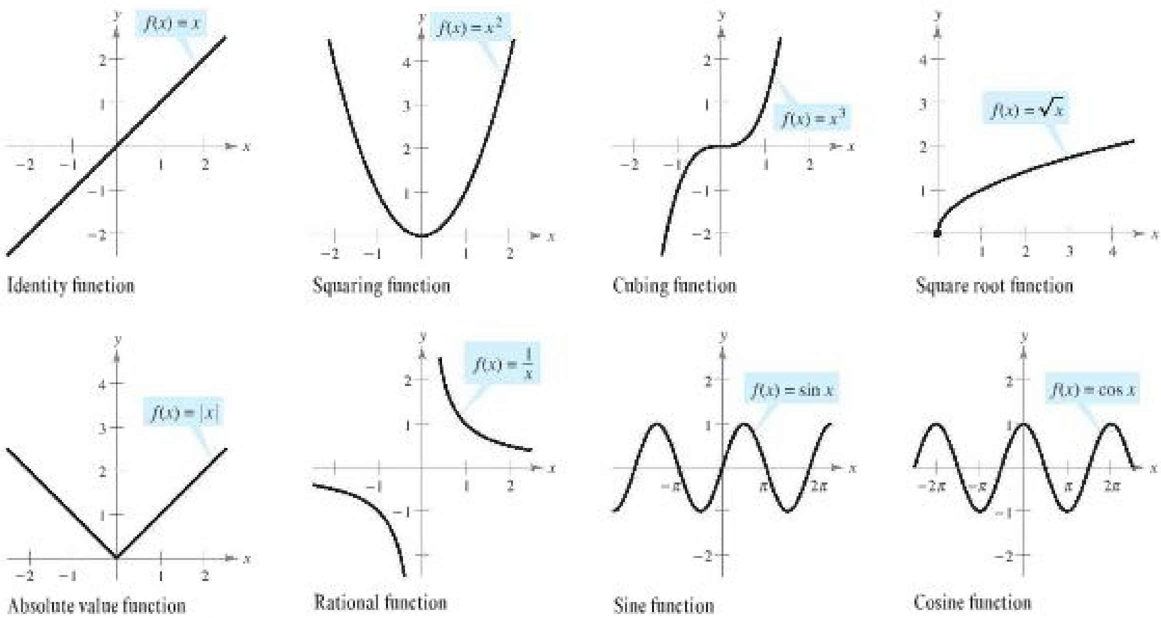
GRAPHING

Below are some basic graphs that you should know:



The graphs of eight basic functions
Figure P.27

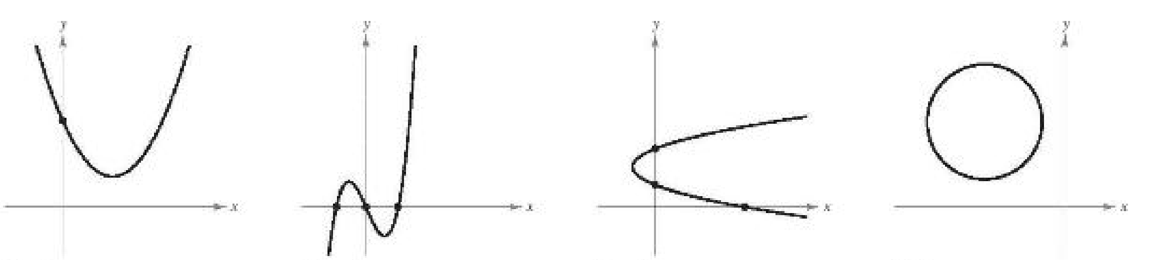
INTERCEPTS

Intercepts of a Graph

Two types of solution points that are especially useful in graphing an equation are those having zero as their x- or y-coordinate. Such points are called **intercepts** because they are the points at which the graph intersects the x- or y-axis. The point $(a, 0)$ is an ***x*-intercept** of the graph of an equation if it is a solution point of the equation. To find the x-intercepts of a graph, let y be zero and solve the equation for x. The point $(0, b)$ is a ***y*-intercept** of the graph of an equation if it is a solution point of the equation. To find the y-intercepts of a graph, let x be zero and solve the equation for y.

NOTE Some texts denote the x-intercept as the x-coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, we will use the term *intercept* to mean either the point or the coordinate.

It is possible for a graph to have no intercepts, or it might have several. For instance, consider the four graphs shown in Figure P.5.



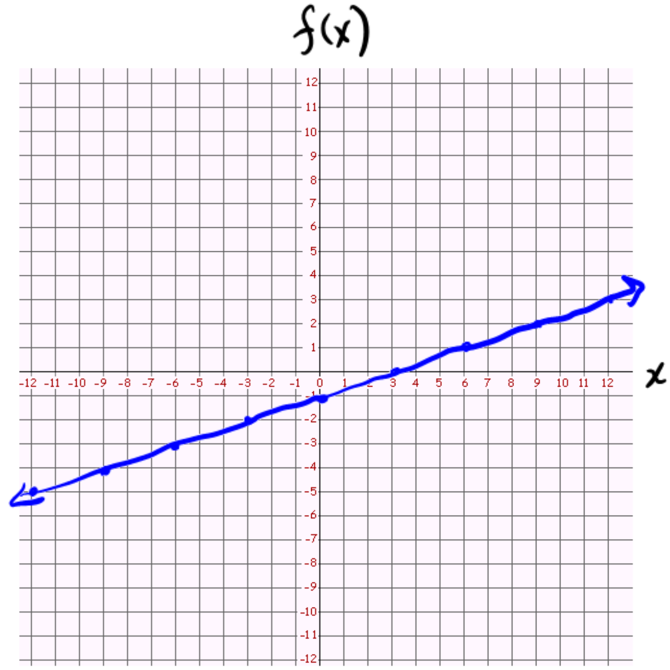
No x-intercepts
One y-intercept
Three x-intercepts
One y-intercept
One x-intercept
Two y-intercepts
No intercepts
Figure P.5

Example 1: Graph the following functions by hand.

a. $f(x)=\frac{x}{3}-1$

$m=\frac{1}{3}$

$b=-1$



b. $g(x)=-x^2+3$

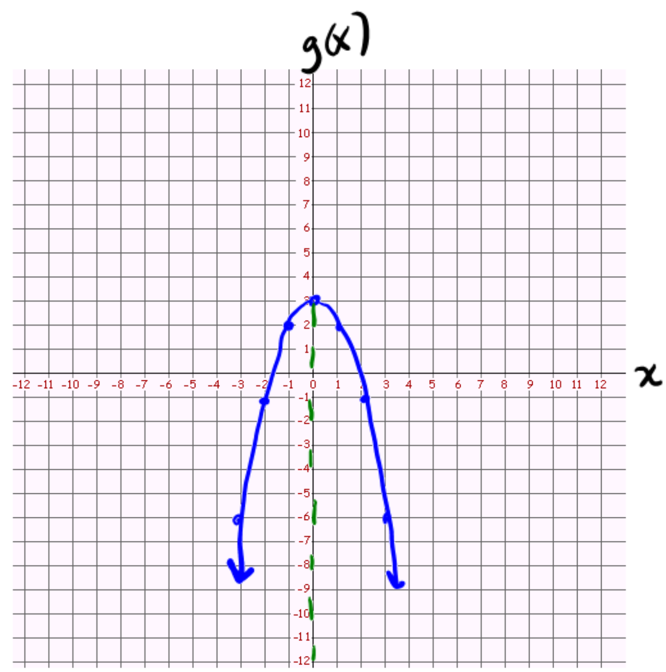
$g(x)=-1(x-0)^2+3$

$g(x)=a(x-h)^2+K$

vertex: $(0,3)$

$g(1)=g(-1)=2$

$g(2)=g(-2)=-1$



SYMMETRY

Symmetry of a Graph

Knowing the symmetry of a graph *before* attempting to sketch it is useful because you need only half as many points to sketch the graph. The following three types of symmetry can be used to help sketch the graphs of equations (see Figure P.7).

1. A graph is **symmetric with respect to the y-axis** if, whenever (x, y) is a point on the graph, $(-x, y)$ is also a point on the graph. This means that the portion of the graph to the left of the y-axis is a mirror image of the portion to the right of the y-axis.
2. A graph is **symmetric with respect to the x-axis** if, whenever (x, y) is a point on the graph, $(x, -y)$ is also a point on the graph. This means that the portion of the graph above the x-axis is a mirror image of the portion below the x-axis.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is a point on the graph, $(-x, -y)$ is also a point on the graph. This means that the graph is unchanged by a rotation of 180° about the origin.

TESTS FOR SYMMETRY

1. The graph of an equation in x and y is symmetric with respect to the y-axis if replacing x by $-x$ yields an equivalent equation.
2. The graph of an equation in x and y is symmetric with respect to the x-axis if replacing y by $-y$ yields an equivalent equation.
3. The graph of an equation in x and y is symmetric with respect to the origin if replacing x by $-x$ and y by $-y$ yields an equivalent equation.

Example 2: Graph the following function by hand. Find any intercepts and test for symmetry.

$y = \sqrt{25 - x^2}$

$y^2 = 25 - x^2$

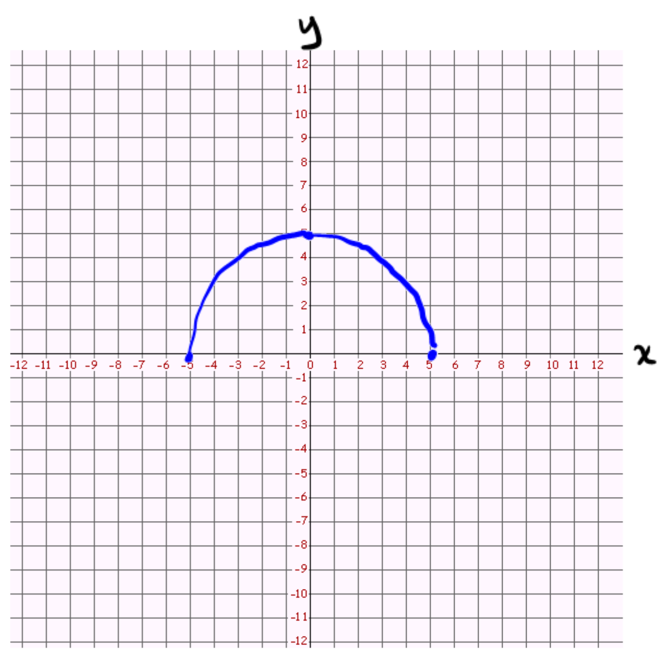
$x^2 + y^2 = 5^2$

only the top half

x-int: $(-5,0), (5,0)$

y-int: $(0,5)$

Symmetric w/respect to the y-axis.



Points of Intersection

A **point of intersection** of the graphs of two equations is a point that satisfies both equations. You can find the point(s) of intersection of two graphs by solving their equations simultaneously.

Example 3: Find the points of intersection of the graphs of the equations.

$x = 3 - y^2$, (A)
$y = x - 1$ (B)

i) Sub. $x = 3 - y^2$ into eq. B

$y = x - 1$

$y = 3 - y^2 - 1$

$y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$

$y+2=0$ or $y-1=0$

$y=-2$ $y=1$

ii) Sub. $y=-2, y=1$ into eq. A

$y=-2$: $x = 3-(-2)^2$
$x=-1$

$y=1$: $x = 3-(1)^2$
$x=2$

iii) Conclusion $\boxed{\{(-1,-2),(2,1)\}, \text{ consistent system}}$

EQUATIONS OF LINES

SUMMARY OF EQUATIONS OF LINES

1.	General form:	$Ax + By + C = 0, \quad (A, B \neq 0)$
2.	Vertical line:	$x = a$
3.	Horizontal line:	$y = b$
4.	Point-slope form:	$y - y_1 = m(x - x_1)$
5.	Slope-intercept form:	$y = mx + b$

Example 4: Find the equation of the line which passes through the given points in point-slope, general, and slope-intercept forms.

a. $(3,-2)$ and $(-8,6)$

$m = \frac{6-(-2)}{-8-3} = -\frac{8}{11}$

$y-(-2) = -\frac{8}{11}(x-3)$

$y+2 = -\frac{8}{11}(x-3)$ point-slope form

$11(y+2) = \left[-\frac{8}{\cancel{11}}(x-3)\right]\cancel{11}$

$11y + 22 = -8x + 24$

$8x + 11y - 2 = 0$ general form

$11y = -8x + 2$

$y = -\frac{8}{11}x + \frac{2}{11}$ Slope-intercept form

b. $\left(\frac{1}{2}, 0\right)$ and $\left(\frac{7}{16}, -\frac{1}{3}\right)$

PARALLEL AND PERPENDICULAR LINES

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal—that is, if and only if $m_1 = m_2$.
2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other—that is, if and only if

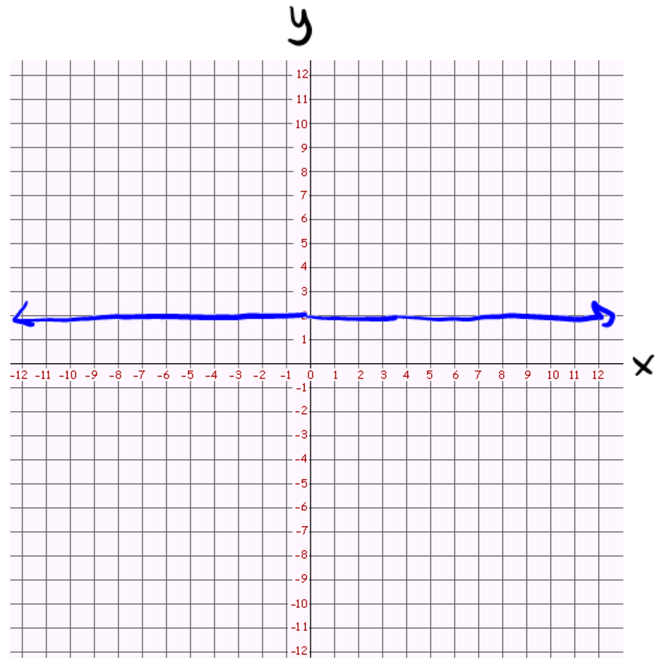
$$m_1 = -\frac{1}{m_2}.$$

Example 5: Graph by hand.

a. $2y = 4$

$y = 2$

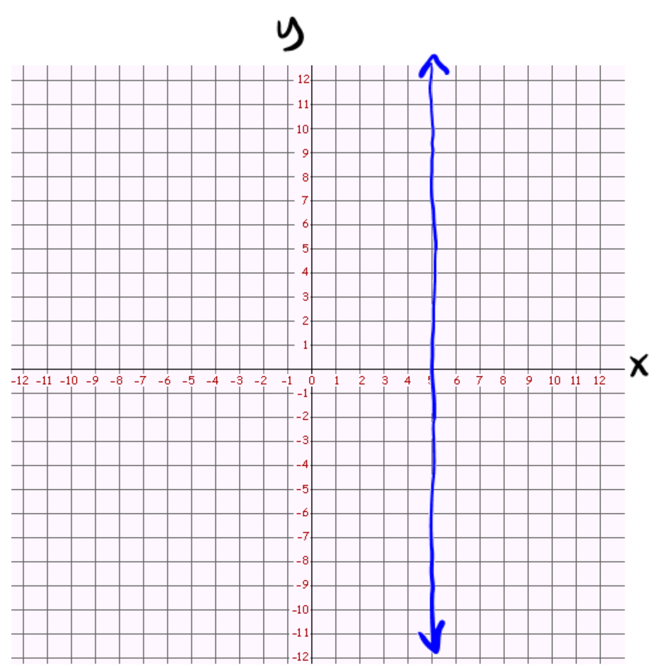
$m = 0$



b. $5 - x = 0$

$x = 5$

m is undefined



FUNCTIONS

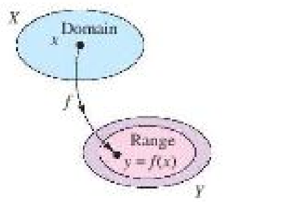
Functions and Function Notation

A **relation** between two sets X and Y is a set of ordered pairs, each of the form (x, y), where x is a member of X and y is a member of Y. A **function** from X to Y is a relation between X and Y that has the property that any two ordered pairs with the same x-value also have the same y-value. The variable x is the **independent variable**, and the variable y is the **dependent variable.**

Many real-life situations can be modeled by functions. For instance, the area A of a circle is a function of the circle's radius r.

$A = \pi r^2$ A is a function of r.

In this case r is the independent variable and A is the dependent variable.



A real-valued function f of a real variable
Figure P.22

DEFINITION OF A REAL-VALUED FUNCTION OF A REAL VARIABLE

Let X and Y be sets of real numbers. A **real-valued function f of a real variable x** from X to Y is a correspondence that assigns to each number x in X exactly one number y in Y.

The **domain** of f is the set X. The number y is the **image** of x under f and is denoted by $f(x)$, which is called the **value of f at x.** The **range** of f is a subset of Y and consists of all images of numbers in X (see Figure P.22).

Example 6: For the function f defined by $f(x) = x^2 - x + 4$, evaluate each expression.

a. $f(-2a) = (-2a)^2 - (-2a) + 4$

$\boxed{f(-2a) = 4a^2 + 2a + 4}$

b. $f(b-1) = (b-1)^2 - (b-1) + 4$

$f(b-1) = b^2 - 2b + 1 - b + 1 + 4$

$\boxed{f(b-1) = b^2 - 3b + 6}$

c. $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}, \ \Delta x \neq 0$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{[(x+\Delta x)^2 - (x+\Delta x) + 4] - [x^2 - x + 4]}{\Delta x}$$

$$= \frac{\cancel{x^2} + 2x\Delta x + \Delta x^2 - \cancel{x} - \Delta x + \cancel{4} - \cancel{x^2} + \cancel{x} - \cancel{4}}{\Delta x}$$

$$= \frac{\cancel{\Delta x}(2x + \Delta x - 1)}{\cancel{\Delta x}}$$

$$= \boxed{2x + \Delta x - 1}$$

The Domain and Range of a Function

The domain of a function can be described explicitly, or it may be described *implicitly* by an equation used to define the function. The implied domain is the set of all real numbers for which the equation is defined, whereas an explicitly defined domain is one that is given along with the function. For example, the function given by

$$f(x) = \frac{1}{x^2 - 4}, \quad 4 \le x \le 5$$

has an explicitly defined domain given by $\{x: 4 \le x \le 5\}$. On the other hand, the function given by

$$g(x) = \frac{1}{x^2 - 4}$$

has an implied domain that is the set $\{x: x \ne \pm 2\}$.

Example 7: Find the domain of the following functions.

$$A^2 - B^2 = (A+B)(A-B)$$
$$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$$
$$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$$

a. $f(x) = \dfrac{x-10}{x^2-100}$

$$f(x) = \frac{\cancel{x-10}}{\cancel{(x-10)}(x+10)} = \frac{1}{x+10}, \; x \ne 10$$

Domain: $\{x : x \ne \pm 10\}$ set-builder

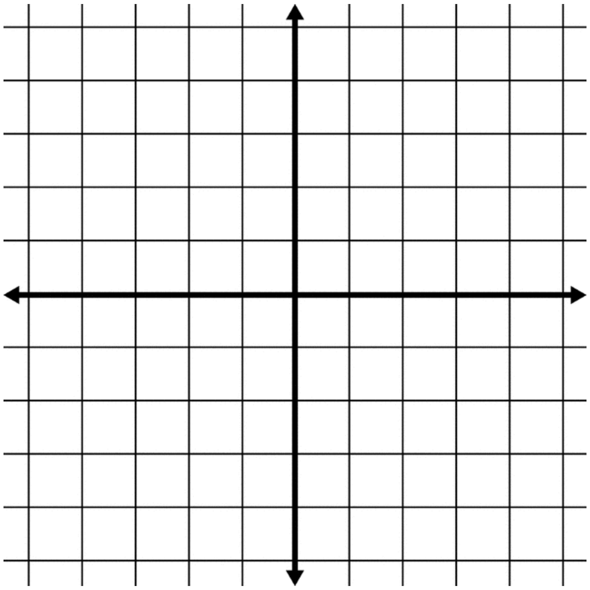
Domain: $(-\infty, -10) \cup (-10, 10) \cup (10, \infty)$

b. $g(x) = \dfrac{x}{x^2 - 5x + 6}$

c. $f(x) = \cot x$

BASIC TYPES OF TRANSFORMATIONS $(c > 0)$	
Original graph:	$y = f(x)$
Horizontal shift c units to the **right:**	$y = f(x - c)$
Horizontal shift c units to the **left:**	$y = f(x + c)$
Vertical shift c units **downward:**	$y = f(x) - c$
Vertical shift c units **upward:**	$y = f(x) + c$
Reflection (about the x-axis):	$y = -f(x)$
Reflection (about the y-axis):	$y = f(-x)$
Reflection (about the origin):	$y = -f(-x)$

Example 8: If $f(x) = \cos x,\ [0, 2\pi)$, graph $f\left(x - \frac{\pi}{2}\right)$



The most common type of algebraic function is a **polynomial function**

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

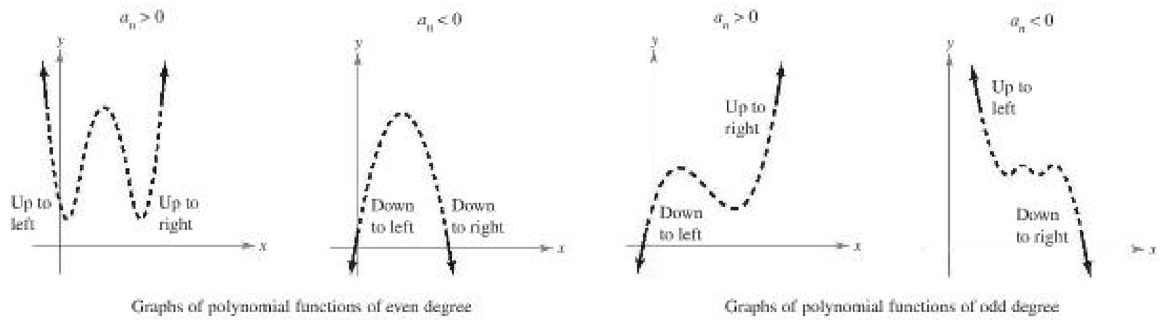
where n is a nonnegative integer. The numbers a_i are **coefficients,** with a_n the **leading coefficient** and a_0 the **constant term** of the polynomial function. If $a_n \neq 0$, then n is the **degree** of the polynomial function. The zero polynomial $f(x) = 0$ is not assigned a degree. It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree, the following simpler forms are often used. (Note that $a \neq 0$.)

Zeroth degree: $f(x) = a$ — Constant function
First degree: $f(x) = ax + b$ — Linear function
Second degree: $f(x) = ax^2 + bx + c$ — Quadratic function
Third degree: $f(x) = ax^3 + bx^2 + cx + d$ — Cubic function

Although the graph of a nonconstant polynomial function can have several turns, eventually the graph will rise or fall without bound as x moves to the right or left. Whether the graph of

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

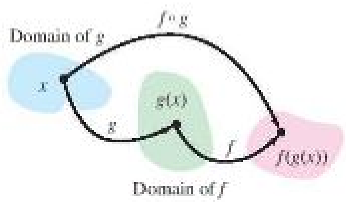
eventually rises or falls can be determined by the function's degree (odd or even) and by the leading coefficient a_n, as indicated in Figure P.29. Note that the dashed portions of the graphs indicate that the **Leading Coefficient Test** determines *only* the right and left behavior of the graph.



The Leading Coefficient Test for polynomial functions
Figure P.29

DEFINITION OF COMPOSITE FUNCTION

Let f and g be functions. The function given by $(f \circ g)(x) = f(g(x))$ is called the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f (see Figure P.30).



The domain of the composite function $f \circ g$
Figure P.30

Example 9: Find $f(x)$ and $g(x)$ if $h(x) = (f \circ g)(x)$ and $h(x) = \sec(5x - 1)$.

$f(x) =$ ____________

$g(x) =$ ____________

TEST FOR EVEN AND ODD FUNCTIONS
The function $y = f(x)$ is **even** if $f(-x) = f(x)$.
The function $y = f(x)$ is **odd** if $f(-x) = -f(x)$.

Example 10: Determine whether $f(x) = x\cos x$ is even, odd, or neither.

Example 11: Find the real zeros of the following functions.

a. $f(x) = 6x^2 - x - 3$ $\rightarrow$ $0 = 6x^2 - x - 3$

-18
-1
not possible!
prime

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(6)(-3)}}{2(6)}$$

$$x = \frac{1 \pm \sqrt{73}}{12}$$

$$\boxed{\left\{ \frac{1-\sqrt{73}}{12}, \frac{1+\sqrt{73}}{12} \right\}}$$

b. $f(x) = (x-8)^2 - 24$

$$0 = (x-8)^2 - 24$$

$$\sqrt{24} = \sqrt{(x-8)^2}$$

$$\pm\sqrt{4 \cdot 6} = x - 8$$

$$\pm 2\sqrt{6} = x - 8$$

$$\rightarrow x = 8 \pm 2\sqrt{6}$$

$$\boxed{\left\{ 8 - 2\sqrt{6}, 8 + 2\sqrt{6} \right\}}$$

c. $g(x) = x^{1/3} + 4$

$$0 = x^{1/3} + 4$$

$$(-4)^3 = \left(x^{1/3}\right)^3$$

$$-64 = x$$

$$\boxed{\{-64\}}$$

d. $h(x) = 2\sin^2 x + \sin x - 1, \ (-\infty, \infty)$

-2

$+2 \quad -1$

$+1$

$0 = 2\sin^2 x + 2\sin x - 1\sin x - 1$

$0 = 2\sin x(\sin x + 1) - 1(\sin x + 1)$

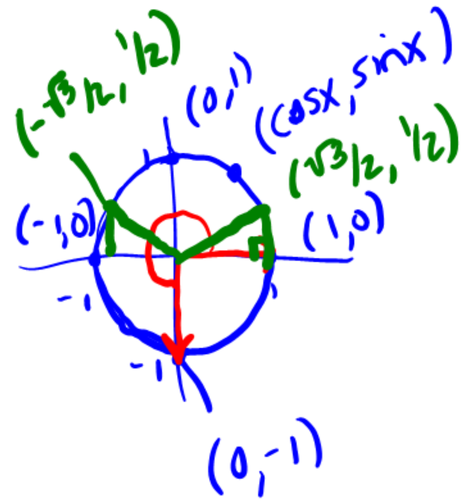
$0 = (\sin x + 1)(2\sin x - 1)$

$\sin x + 1 = 0$ or $2\sin x - 1 = 0$

$\sin x = -1$ $\qquad$ $2\sin x = 1$

$x = \frac{3\pi}{2}$ $\qquad$ $\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$



$x = \left\{ x : x = \frac{\pi}{6} + \frac{2\pi n}{3}, n \in \mathbb{Z} \right\}$

e. $f(x) = 3\tan^2 2x - 1, \ [0, \pi)$